

# Quantum Mechanics – I

Project Assignment

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## Landau Levels and the Integer Quantum Hall Effect

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### Abstract

We provide an independent derivation of Landau levels and their use in describing the IQHE. Beginning from classical mechanics, the Lagrangian and Hamiltonian for a charged particle in crossed  $E$  and  $B$  fields are calculated, with careful consideration of the effects of gauge transformation and freedom.

Quantisation is then performed in the symmetric and Landau gauges. We show that the momentum operators of the system form Heisenberg-like commutators such that the problem reduces to a one-dimensional harmonic oscillator. This yields the energy eigenvalues as

$$E_n = \hbar\omega_c \left( n + \frac{1}{2} \right)$$

in terms of equally spaced and infinitely degenerate Landau levels.

We normalise the lowest energy level wave functions in the symmetric gauge and define the magnetic length of the system as

$$\ell_B = \sqrt{\frac{\hbar c}{eB}}.$$

Degeneracies are calculated from three different perspectives: boundary conditions, the uncertainty principle, and classical cyclotron motion. We consider the effects of the Zeeman term and describe the guiding-centre algebra that gives rise to degeneracy.

In Part 2, we apply these techniques to a problem in the Landau gauge, solve for the harmonic oscillator, and calculate the degeneracy of ground states due to boundary conditions on a finite-sized sample,

$$N = \frac{BA}{\Phi_0}, \quad \text{where} \quad \Phi_0 = \frac{hc}{e}.$$

In the end, we obtain the classical Hall effect, discuss the conductivity and resistivity tensors based on symmetry considerations (Onsager relations), and explain how the quantised filling of Landau levels leads to the famous quantised Hall plateaux

$$\rho_{xy} = \frac{h}{\nu e^2}.$$

Finally, we briefly explain the fractional quantum Hall effect and Laughlin's wave function.

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# 1 Classical Mechanics Framework

## 1.1 Setup and Basic Quantities

Consider an electron of mass  $m$  and charge  $-e$  ( $e > 0$ ) moving in a uniform magnetic field  $\vec{B} = -B\hat{z}$  (pointing into the plane) and electric field  $\vec{E}$ . The Lorentz force law gives the equation of motion:

$$m\ddot{\vec{r}} = -e\left(\vec{E} + \frac{\dot{\vec{r}} \times \vec{B}}{c}\right). \quad (1)$$

### Classical Results

- **Cyclotron radius:**  $R = \frac{mvc}{eB}$
- **Cyclotron frequency:**  $\omega_c = \frac{eB}{mc}$ , equivalently  $\omega_c = \frac{v}{R}$

The electron traces a circular (cyclotron) orbit in the  $xy$ -plane, gyrating at frequency  $\omega_c$ , which is independent of the electron's speed — a key classical result that survives quantisation.

## 1.2 Deriving the Lagrangian

The electromagnetic fields can be expressed in terms of the scalar and vector potentials as:

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}. \quad (2)$$

The Lagrangian  $L$  is given by  $L = T - U$ , where  $T$  is the kinetic energy and  $U$  is the potential energy.

**Free particle.** With no fields,  $L = \frac{1}{2}m\dot{r}^2$ .

**Electrostatic field only.** Adding an electrostatic potential,  $L = \frac{1}{2}m\dot{r}^2 - e\phi$  (only electric field present). The Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (3)$$

gives

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{\partial L}{\partial r} = -e\nabla\phi, \quad (4)$$

so that  $m\ddot{r} = -e\nabla\phi$ , which is the correct equation of motion for an electron in an electric field.

**Adding the magnetic force.** We seek a Lagrangian that reproduces the magnetic force  $\mathbf{F}_{\text{mag}} = \frac{e}{c} \dot{\mathbf{r}} \times \mathbf{B}$ ,  $\mathbf{B} = \nabla \times \mathbf{A}$ . Consider the  $x$ -component:

$$F_x = \frac{e}{c} (\dot{y}B_z - \dot{z}B_y).$$

Using  $\mathbf{B} = \nabla \times \mathbf{A}$ ,

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}, \quad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x},$$

so that

$$F_x = \frac{e}{c} \left[ \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} - \dot{y} \frac{\partial A_x}{\partial y} - \dot{z} \frac{\partial A_x}{\partial z} \right].$$

Noting that  $\dot{\mathbf{r}} \cdot \mathbf{A} = \dot{x}A_x + \dot{y}A_y + \dot{z}A_z$  and

$$\frac{dA_x}{dt} = \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z},$$

we identify

$$F_x = \frac{e}{c} \left[ \frac{\partial}{\partial x} (\dot{\mathbf{r}} \cdot \mathbf{A}) - \frac{dA_x}{dt} \right] = \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right),$$

which is satisfied by adding the linear velocity term  $L_{\text{mag}} = \frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}$ . Hence the full Lagrangian is:

$$L = \frac{1}{2} m |\dot{\mathbf{r}}|^2 + \frac{e}{c} \dot{\mathbf{r}} \cdot \vec{A} - e\phi. \quad (5)$$

**Verification via Euler–Lagrange.** Using the Euler–Lagrange equation along the  $x$ -component:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{e}{c} A_x, \quad (6)$$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = m\ddot{x} + \frac{e}{c} \left( \frac{\partial A_x}{\partial t} + \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z} \right), \quad (7)$$

$$\frac{\partial L}{\partial x} = \frac{e}{c} \left( \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} \right) - e \frac{\partial \phi}{\partial x}. \quad (8)$$

Setting  $\frac{\partial L}{\partial x} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right)$  and simplifying using  $B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$ ,  $B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$ , we recover:

$$m\ddot{x} = -eE_x + \frac{e}{c} (\dot{y}B_z - \dot{z}B_y) = -eE_x + \frac{e}{c} (\dot{\mathbf{r}} \times \vec{B})_x, \quad (9)$$

confirming the Lagrangian is correct since the equation of motion is the same.

### 1.3 Canonical Momentum and Hamiltonian

The **canonical momentum** conjugate to  $q_i$  is  $p_i = \partial L / \partial \dot{q}_i$ :

$$\boxed{p_i = m\dot{x}_i + \frac{e}{c}A_i, \quad \Rightarrow \quad m\dot{\vec{r}} = \vec{p} - \frac{e}{c}\vec{A}.} \quad (10)$$

Note that  $\vec{p}$  is the *canonical* momentum, while  $m\dot{\vec{r}} \equiv \vec{\Pi}$  is the *kinetic* (mechanical) momentum. These are distinct whenever  $\vec{A} \neq 0$ .

Substituting  $\dot{\vec{r}} = \frac{1}{m}(p - \frac{e}{c}A)$  back into the Lagrangian:

$$\frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2m}\left|p - \frac{e}{c}A\right|^2, \quad \frac{e}{c}\dot{\vec{r}} \cdot \vec{A} = \frac{e}{mc}\left(p - \frac{e}{c}A\right) \cdot \vec{A}.$$

**Legendre Transform.** The Hamiltonian is obtained via  $H = \sum_i p_i \dot{q}_i - L$ :

$$\begin{aligned} H &= \vec{p} \cdot \dot{\vec{r}} - L = \frac{1}{m}\left|\vec{p} - \frac{e}{c}\vec{A}\right|^2 + \frac{e}{mc}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \cdot \vec{A} - \frac{1}{2m}\left|\vec{p} - \frac{e}{c}\vec{A}\right|^2 - \frac{e}{c}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \cdot \frac{\vec{A}}{m} + e\phi \\ \Rightarrow \quad &\boxed{H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi.} \end{aligned} \quad (11)$$

This is precisely the minimal coupling Hamiltonian (Griffiths, Eq. 4.190). Hamilton's equations of motion are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right), \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{e}{mc}\left(\vec{p} - \frac{e}{c}\vec{A}\right) \cdot \frac{\partial \vec{A}}{\partial q_i} - e\frac{\partial \phi}{\partial q_i}. \quad (12)$$

## 2 Gauge Theory: Freedom and Transformations

### 2.1 Multiple Gauges, Same Physics

Since  $\vec{B} = \nabla \times \vec{A}$ , the vector potential is not unique: *multiple choices of  $\vec{A}$  yield the same  $\vec{B}$* . Specifically, if  $\vec{A}$  gives  $\vec{B}$ , then so does  $\vec{A}' = \vec{A} + \nabla\chi(x, y, z, t)$  for any scalar function  $\chi$ .

**Proof.**  $\nabla \times \vec{A}' = \nabla \times \vec{A} + \nabla \times (\nabla\chi) = \vec{B} + 0 = \vec{B}$ , since the curl of a gradient always vanishes. This is **Helmholtz's theorem**.

A particular choice of  $\vec{A}$  is called a **gauge**. Both  $\phi$  and  $\vec{A}$  have gauge freedom; the full transformation is:

$$\vec{A}' = \vec{A} + \nabla\chi, \quad \phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}. \quad (13)$$

In our problem  $\phi = 0$ , so the scalar potential transformation can be ignored.

**Note**

The general gauge family for a uniform field  $\vec{B} = B\hat{z}$  is  $A_x = -\alpha By$ ,  $A_y = (1 - \alpha)Bx$ , for free parameter  $\alpha$ . One can verify  $\nabla \times \vec{A} = B\hat{z}$  for any  $\alpha$ .

**2.2 Why a Gauge Cannot Satisfy All Symmetries**

The problem has (i) **translational symmetry** in  $x$  and  $y$  (the field is uniform), and (ii) **rotational symmetry** about the  $z$ -axis. However, it is mathematically impossible for a single gauge to simultaneously respect all symmetries, because  $\nabla \times (\text{const. vector}) = 0 \neq \vec{B}$ , so a constant vector potential cannot reproduce  $\vec{B}$ . The translational symmetry requires a non-constant  $\vec{A}$ , inevitably breaking some rotational covariance at the level of  $\vec{A}$  (though not of the physical fields).

**2.3 Symmetric Gauge**

$$\vec{A} = \frac{B}{2}(-y, x, 0), \quad A_x = -\frac{By}{2}, \quad A_y = +\frac{Bx}{2}. \quad (14)$$

**Verification.**

$$(\nabla \times \vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \frac{B}{2} + \frac{B}{2} = B. \quad \checkmark$$

If instead  $\vec{B} = -B\hat{z}$ , then  $\vec{A} = \frac{B}{2}(y, -x, 0)$ .

**Rotational symmetry.** Under a rotation  $x \rightarrow x \cos \theta - y \sin \theta$ ,  $y \rightarrow x \sin \theta + y \cos \theta$ , one finds  $A'_x = A_x \cos \theta - A_y \sin \theta$  and  $A'_y = A_x \sin \theta + A_y \cos \theta$ , i.e.  $\vec{A}$  transforms as a vector — the gauge is *rotationally covariant*.  $\vec{A}$  is always tangential and depends only on distance from the origin.

**2.4 Landau Gauge**

$$\vec{A} = (0, Bx, 0) \quad \text{or} \quad \vec{A} = (-By, 0, 0). \quad (15)$$

This preserves translational symmetry along  $y$  (or  $x$ ). If  $\vec{A} = (0, Bx, 0)$  and  $y \rightarrow y + a$ , then  $\vec{A} \rightarrow \vec{A}$  unchanged (since it depends only on  $x$ ). However, shifting  $x \rightarrow x + a$  sends  $A_y = Bx \rightarrow B(x + a) \neq A_y$ .

**Relationship between gauges.** The symmetric and Landau gauges differ by a gradient term. Writing  $\mathbf{A}_S = \frac{B}{2}(-y, x, 0)$  and  $\mathbf{A}_L = (0, Bx, 0)$ :

$$\mathbf{A}_S - \mathbf{A}_L = \left( -\frac{B}{2}y, \frac{B}{2}x - Bx, 0 \right) = \left( -\frac{B}{2}y, -\frac{B}{2}x, 0 \right) = \nabla \left( -\frac{B}{2}xy \right) = \nabla \chi.$$

Hence the two gauges are related by a gauge transformation with  $\chi = -\frac{B}{2}xy$ .

### 3 Quantum Mechanical Formulation: Symmetric Gauge

#### 3.1 The Schrödinger Equation with Minimal Coupling

Canonical quantisation replaces  $\vec{p} \rightarrow -i\hbar\nabla$ . With  $H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 + e\phi$ , the time-dependent Schrödinger equation becomes:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - \frac{e}{c}\vec{A}\right)^2\Psi + e\phi\Psi.} \quad (16)$$

This is the quantum implementation of the Lorentz force law, sometimes called the *minimal coupling rule* (Griffiths §4.5).

For the symmetric gauge  $\vec{A} = \frac{B}{2}(-y, x, 0)$ :

#### 3.2 Kinetic Momentum Operators

It is convenient to define the **kinetic momentum operators**:

$$\boxed{\Pi_x = p_x - \frac{e}{c}A_x, \quad \Pi_y = p_y - \frac{e}{c}A_y, \quad \Pi_z = p_z - \frac{e}{c}A_z.} \quad (17)$$

Substituting the symmetric gauge ( $A_x = -By/2$ ,  $A_y = Bx/2$ ,  $A_z = 0$ ):

$$\Pi_x = p_x + \frac{eB}{2c}y, \quad \Pi_y = p_y - \frac{eB}{2c}x, \quad \Pi_z = p_z. \quad (18)$$

#### 3.3 Commutation Relations of $\Pi_x$ and $\Pi_y$

This is the central computation. We compute  $[\Pi_x, \Pi_y]$ :

$$[\Pi_x, \Pi_y] = \left[ p_x + \frac{eB}{2c}y, p_y - \frac{eB}{2c}x \right] \quad (19)$$

$$= [p_x, p_y] - \frac{eB}{2c}[p_x, x] + \frac{eB}{2c}[y, p_y] - \left(\frac{eB}{2c}\right)^2 [y, x]. \quad (20)$$

Different components of  $\vec{p}$  commute, so  $[p_x, p_y] = 0$ . Also  $[y, x] = 0$ . Using  $[p_x, x] = -i\hbar$  and  $[y, p_y] = i\hbar$ :

$$[\Pi_x, \Pi_y] = -\frac{eB}{2c}(-i\hbar) + \frac{eB}{2c}(i\hbar) = \frac{ieB\hbar}{2c} + \frac{ieB\hbar}{2c} = \frac{ieB\hbar}{c}. \quad (21)$$

#### Key Result

$$[\Pi_x, \Pi_y] = \frac{ieB\hbar}{c} = im\hbar\omega_c. \quad (22)$$

**Commutators involving  $\Pi_z$ .** Since  $\Pi_z = p_z$  (the  $z$ -component of  $\vec{A}$  is zero in the symmetric gauge), we verify:

$$[\Pi_z, \Pi_x] = \left[ p_z, p_x + \frac{eB}{2c}y \right] = [p_z, p_x] + \frac{eB}{2c}[p_z, y] = 0 + 0 = 0, \quad (23)$$

since  $[p_z, p_x] = 0$  (different momentum components commute) and  $[p_z, y] = -i\hbar \partial_z(y) = 0$  ( $y$  has no  $z$ -dependence). Similarly  $[\Pi_z, \Pi_y] = 0$ . Therefore  $\Pi_z$  **perfectly commutes** with both  $\Pi_x$  and  $\Pi_y$ .

**Compact notation for the Hamiltonian.** With all cross-commutators established, the Hamiltonian takes the elegant form:

$$H = \frac{1}{2m} (\Pi_x^2 + \Pi_y^2 + \Pi_z^2). \quad (24)$$

### Important Concept: Commuting Operators Share Eigenstates

If two operators  $\hat{A}$  and  $\hat{B}$  *perfectly commute*,  $[\hat{A}, \hat{B}] = 0$ , they *share common eigenstates*.

**Proof:** If  $\hat{A}\Psi = a\Psi$ , then  $\hat{A}(\hat{B}\Psi) = \hat{B}(\hat{A}\Psi) = a(\hat{B}\Psi)$ , so  $\hat{B}\Psi$  is also an eigenstate of  $\hat{A}$  with the same eigenvalue  $a$ .

**Application:** Since  $\Pi_z = p_z$  and  $[H, p_z^2] = 0$ :

$$\left[ H, \frac{p_z^2}{2m} \right] = \left[ \frac{\Pi_x^2 + \Pi_y^2 + p_z^2}{2m}, \frac{p_z^2}{2m} \right] = 0, \quad (25)$$

so we can simultaneously diagonalise  $H$  and  $p_z^2$ , and the energy separates as  $E = E_{xy} + E_z$ .

### 3.4 Free Particle Analogy: The $z$ -Direction

Since  $\Pi_z = p_z$  and  $[H, p_z^2] = 0$ , the  $z$ -direction behaves as a completely **free particle**. The  $H_z$  part of the Hamiltonian is:

$$H_z = \frac{p_z^2}{2m} = -\frac{\hbar^2}{2m} \frac{d^2}{dz^2}, \quad (26)$$

which is identical in form to the Hamiltonian of a free particle,  $H_{\text{free}} = p^2/2m$ .

**Solving the free-particle eigenvalue problem.** To find eigenvalues and eigenstates, solve:

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dz^2} = E_z\psi \implies \frac{d^2\psi}{dz^2} = -k^2\psi, \quad k = \sqrt{\frac{2mE_z}{\hbar^2}}, \quad (27)$$

giving:

$$\boxed{E_z = \frac{\hbar^2 k_z^2}{2m}, \quad \psi(z) = Ae^{ik_z z} + Be^{-ik_z z}.} \quad (28)$$

**Momentum eigenstates.**  $\psi_k = e^{ikz}$  is an eigenstate of  $\hat{p}_z = -i\hbar\partial_z$ :

$$\hat{p}_z \psi_k = -i\hbar \frac{\partial}{\partial z} e^{ikz} = \hbar k \cdot e^{ikz}. \quad (29)$$

#### Why we don't take two separate wave functions

We do not separately include  $e^{-ikz}$  because letting  $k \in (-\infty, \infty)$  automatically covers  $e^{-ikz}$  (it is  $e^{ikz}$  with  $k \rightarrow -k$ ). A single family  $\psi_k = e^{ikz}$ ,  $k \in \mathbb{R}$ , forms a complete set.

**Non-normalisability and Dirac normalisation.** These eigenstates satisfy:  $\langle k'|k \rangle = \int_{-\infty}^{\infty} e^{-ik'z} e^{ikz} dz = 2\pi \delta(k - k')$ , so one uses the Dirac-normalised form  $\psi_k(z) = \frac{1}{\sqrt{2\pi}} e^{ikz}$ .

**Energy splitting.** The full wave function factorises as  $\Psi(x, y, z) = \Psi_{xy}(x, y) \cdot e^{ik_z z}$ , and:

$$\boxed{E = E_{xy} + E_z = E_{xy} + \frac{\hbar^2 k_z^2}{2m}.} \quad (30)$$

The non-trivial quantisation lives entirely in  $E_{xy}$ , solved by the harmonic oscillator analogy below.

### 3.5 Hamiltonian and the Harmonic Oscillator Analogy

The Hamiltonian separates as  $H = H_{xy} + H_z$  with:

$$H_{xy} = \frac{\Pi_x^2 + \Pi_y^2}{2m}, \quad H_z = \frac{\Pi_z^2}{2m} = \frac{p_z^2}{2m}. \quad (31)$$

Since  $[\Pi_z, H_{xy}] = 0$ , the two parts share simultaneous eigenstates and the energy splits:  $E = E_{xy} + E_z$ .

**Canonically conjugate  $\Pi$  operators.** Although  $\Pi_x$  and  $\Pi_y$  do not commute, their commutator is a  $c$ -number. We can rescale them to obtain a standard canonical pair. Using bilinearity:

$$[\lambda\Pi_x, \lambda\Pi_y] = \lambda^2 [\Pi_x, \Pi_y] = \lambda^2 \cdot \frac{ieB\hbar}{c}. \quad (32)$$

Choosing  $\lambda = \sqrt{c/(eB)}$ :

$$\left[ \sqrt{\frac{c}{eB}} \Pi_x, \sqrt{\frac{c}{eB}} \Pi_y \right] = \frac{c}{eB} \cdot \frac{ieB\hbar}{c} = i\hbar. \quad (33)$$

### Canonically Conjugate Pair

$$\left[ \sqrt{\frac{c}{eB}} \Pi_x, \sqrt{\frac{c}{eB}} \Pi_y \right] = i\hbar. \quad (34)$$

The rescaled kinetic momenta are **canonically conjugate** — they satisfy the same commutation relation as  $x$  and  $p$ . This is the precise sense in which the 2D problem maps onto a 1D harmonic oscillator.

### Physical Insight

Equation (22) shows that  $[\Pi_x, \Pi_y] \propto i\hbar$ , structurally identical to  $[x, p] = i\hbar$ . So  $\Pi_x$  and  $\Pi_y$  cannot be treated as ordinary numbers; one builds ladder operators from them, exactly as in the harmonic oscillator. The canonical conjugate pair above makes this analogy completely rigorous.

## 3.6 Ladder Operators

Define:

$$b = \sqrt{\frac{c}{2eB\hbar}} (\Pi_x + i\Pi_y), \quad b^\dagger = \sqrt{\frac{c}{2eB\hbar}} (\Pi_x - i\Pi_y). \quad (35)$$

These are chosen so that  $[b, b^\dagger] = 1$ :

$$[b, b^\dagger] = \frac{c}{2eB\hbar} [\Pi_x + i\Pi_y, \Pi_x - i\Pi_y] = \frac{c}{2eB\hbar} (-i[\Pi_x, \Pi_y] + i[\Pi_y, \Pi_x]) = \frac{c}{2eB\hbar} \cdot 2 \cdot \frac{eB\hbar}{c} = 1. \quad \checkmark \quad (36)$$

We also have  $bb^\dagger = b^\dagger b + 1$ .

Inverting to express  $\Pi_x, \Pi_y$  in terms of  $b, b^\dagger$ :

$$\Pi_x = \frac{1}{2} \sqrt{\frac{2eB\hbar}{c}} (b + b^\dagger), \quad \Pi_y = \frac{1}{2i} \sqrt{\frac{2eB\hbar}{c}} (b - b^\dagger). \quad (37)$$

Then:

$$\Pi_x^2 = \frac{eB\hbar}{2c} (b^2 + (b^\dagger)^2 + bb^\dagger + b^\dagger b), \quad (38)$$

$$\Pi_y^2 = \frac{eB\hbar}{2c} \cdot (-1) (b^2 + (b^\dagger)^2 - bb^\dagger - b^\dagger b), \quad (39)$$

$$\Pi_x^2 + \Pi_y^2 = \frac{eB\hbar}{c} (bb^\dagger + b^\dagger b) = \frac{2eB\hbar}{c} \left( b^\dagger b + \frac{1}{2} \right). \quad (40)$$

Therefore:

$$H_{xy} = \frac{eB\hbar}{mc} \left( b^\dagger b + \frac{1}{2} \right) = \hbar\omega_c \left( b^\dagger b + \frac{1}{2} \right). \quad (41)$$

### 3.7 Landau Levels: Energy Spectrum

Since  $b^\dagger b$  has non-negative integer eigenvalues  $n = 0, 1, 2, \dots$ :

$$E_n = \hbar\omega_c \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \dots \quad (42)$$

These are the **Landau levels**. The  $n$ -th excited state is:

$$|n, m\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}} |0, m\rangle, \quad H |n, m\rangle = \hbar\omega_c \left( n + \frac{1}{2} \right) |n, m\rangle. \quad (43)$$

The label  $m$  encodes the massive *degeneracy*, further derivation on the above formula is done later (see Section 8).

## 4 Zeeman Effect

The Zeeman effect refers to the splitting of energy levels when an electron's spin magnetic moment aligns with or against the external field.

The spin magnetic moment is:

$$\vec{\mu}_S = -g \frac{e}{2mc} \vec{S}, \quad S_z = m_s \hbar, \quad m_s = \pm \frac{1}{2}, \quad (44)$$

where  $g = 2$  for a free electron (from Dirac theory). For a magnetic field  $\vec{B} = B\hat{z}$ :

$$H_{\text{Zeeman}} = -\vec{\mu}_S \cdot \vec{B} = -\mu_{S,z} B, \quad \mu_{S,z} = -\frac{ge}{2mc} S_z = -\frac{ge}{2mc} m_s \hbar. \quad (45)$$

$$H_{\text{Zeeman}} = gm_s \mu_B B, \quad \mu_B = \frac{e\hbar}{2mc} \text{ (Bohr magneton)}. \quad (46)$$

This gives:

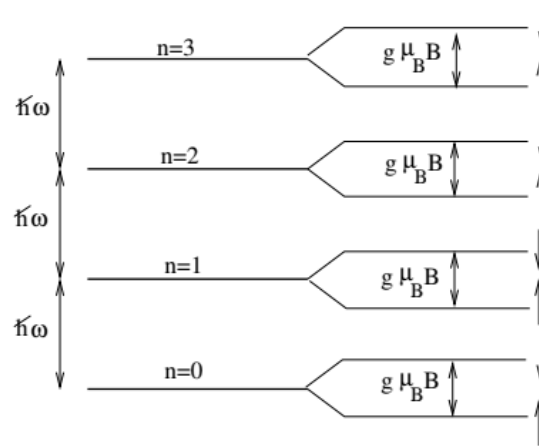
$$E_{\text{spin up}} = +\frac{1}{2} g \mu_B B, \quad (47)$$

$$E_{\text{spin down}} = -\frac{1}{2} g \mu_B B. \quad (48)$$

### 4.1 Combined Spectrum

The full energy combining both effects is:

$$E_{n,m_s} = \hbar\omega_c \left( n + \frac{1}{2} \right) + gm_s \mu_B B. \quad (49)$$



For free electrons  $g\mu_B B = \frac{ge\hbar B}{2mc} = \frac{e\hbar B}{mc} = \hbar\omega_c$ , so:

$$E_{n,\uparrow} = \hbar\omega_c \left( n + \frac{1}{2} \right) + \frac{1}{2} \hbar\omega_c = \hbar\omega_c (n + 1), \quad (50)$$

$$E_{n,\downarrow} = \hbar\omega_c \left( n + \frac{1}{2} \right) - \frac{1}{2} \hbar\omega_c = \hbar\omega_c n. \quad (51)$$

The spin-up  $n$ -th and spin-down  $(n + 1)$ -th levels become degenerate; Landau levels including Zeeman splitting interleave with spacing  $\hbar\omega_c$  between consecutive sub-levels.

## 5 Ground State Wave Function

### 5.1 Finding the Ground State

Every ladder operator formalism requires a ground state  $|0\rangle$  satisfying  $b|0\rangle = 0$ :

$$b|0\rangle = 0 \implies \sqrt{\frac{c}{2eB\hbar}} (\Pi_x + i\Pi_y) |0\rangle = 0. \quad (52)$$

With  $p_x = -i\hbar\partial_x$ ,  $p_y = -i\hbar\partial_y$  and  $A_x = -By/2$ ,  $A_y = Bx/2$  (CGS):

$$(\Pi_x + i\Pi_y)\Psi_0 = \left( -i\hbar\partial_x + \frac{eB}{2c}y + i \left( -i\hbar\partial_y - \frac{eB}{2c}x \right) \right) \Psi_0 = 0, \quad (53)$$

$$\left[ -i\hbar\frac{\partial}{\partial x} + \hbar\frac{\partial}{\partial y} + \frac{eB}{2c}(y - ix) \right] \Psi_0 = 0. \quad (54)$$

### 5.2 Complex Coordinate Trick

Define complex coordinates  $z = x + iy$  and  $\bar{z} = x - iy$ . Using the chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial f}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial y}. \quad (55)$$

Since  $z = x + iy$  and  $\bar{z} = x - iy$ , we have  $\partial z/\partial x = 1$ ,  $\partial \bar{z}/\partial x = 1$ ,  $\partial z/\partial y = i$ ,  $\partial \bar{z}/\partial y = -i$ , giving:

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, & \frac{\partial}{\partial y} &= i \frac{\partial}{\partial z} - i \frac{\partial}{\partial \bar{z}}, \\ \frac{\partial}{\partial z} &= \frac{1}{2} \frac{\partial}{\partial x} - \frac{i}{2} \frac{\partial}{\partial y}, & \frac{\partial}{\partial \bar{z}} &= \frac{1}{2} \frac{\partial}{\partial x} + \frac{i}{2} \frac{\partial}{\partial y}.\end{aligned}$$

Combining the operator expressions:

$$-i\hbar \frac{\partial}{\partial x} + \hbar \frac{\partial}{\partial y} = -i\hbar \left( \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) + i\hbar \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) = -2i\hbar \frac{\partial}{\partial \bar{z}}.$$

Also  $y - ix = -i(x + iy) = -iz$ , so combining everything the ground-state equation becomes:

$$\left( -2i\hbar \frac{\partial}{\partial \bar{z}} - \frac{ieB}{2c} z \right) \Psi_0 = 0 \implies \left( \frac{\partial}{\partial \bar{z}} + \frac{eB}{4\hbar c} z \right) \Psi_0 = 0. \quad (56)$$

This is a first-order PDE. The ansatz  $\Psi_0 = \exp(-\frac{eB}{4\hbar c} z \bar{z}) f(z, \bar{z})$  gives:

$$\frac{\partial f}{\partial \bar{z}} = 0, \quad (57)$$

so  $f$  is an **analytic (holomorphic) function** independent of  $\bar{z}$ . For the ground state:  $f(z) = z^m$ , analogous to Hermite polynomials in the harmonic oscillator.

### Ground State Wave Functions

$$\Psi_0^{(m)}(z, \bar{z}) = \mathcal{N}_m z^m \exp\left(-\frac{eB}{4\hbar c} |z|^2\right), \quad m = 0, 1, 2, \dots \quad (58)$$

The integer  $m \geq 0$  labels the *degeneracy* within the ground Landau level. Multiple values of  $m$  give the same energy  $E_0 = \frac{1}{2} \hbar \omega_c$  but different orbital shapes/locations.

### 5.3 Normalisation

We require  $\int |\Psi_0^{(m)}|^2 d^2r = 1$ . In polar coordinates  $z = re^{i\theta}$ :

$$|\mathcal{N}_m|^2 \int_0^{2\pi} \int_0^\infty r^{2m} \exp\left(-\frac{eB}{2\hbar c} r^2\right) r dr d\theta = 1. \quad (59)$$

Let  $\alpha = eB/(2\hbar c)$  and substitute  $u = \alpha r^2$ ,  $du = 2\alpha r dr$ :

$$I = \int_0^\infty r^{2m} e^{-\alpha r^2} r dr = \int_0^\infty \left(\frac{u}{\alpha}\right)^m e^{-u} \frac{du}{2\alpha} = \frac{1}{2\alpha^{m+1}} \int_0^\infty u^m e^{-u} du = \frac{\Gamma(m+1)}{2\alpha^{m+1}} = \frac{m!}{2\alpha^{m+1}}.$$

So  $2\pi |\mathcal{N}_m|^2 \cdot \frac{m!}{2\alpha^{m+1}} = 1$ , giving:

## Key Result

$$\mathcal{N}_m = \left( \pi m! \left( \frac{2\hbar c}{eB} \right)^{m+1} \right)^{-1/2}. \quad (60)$$

## 6 The Magnetic Length and Physical Interpretation of $m$

### 6.1 Definition

$$\ell_B = \sqrt{\frac{\hbar c}{eB}}. \quad (61)$$

This is the fundamental length scale of the problem.

### 6.2 Three Ways to Derive the Magnetic Length

(1) **From the wave function directly.** The ground state ( $m = 0$ ) is  $\Psi_0^{(0)} \propto e^{-|z|^2/(4\ell_B^2)}$ , a Gaussian of width  $\ell_B$ . Thus  $\ell_B$  is the spatial extent of the ground-state cyclotron orbit.

(2) **From the uncertainty principle (Cauchy–Schwarz).** Since  $[\Pi_x, \Pi_y] = ieB\hbar/c$ , the generalised uncertainty relation gives:

$$\Delta\Pi_x \cdot \Delta\Pi_y \geq \frac{eB\hbar}{2c}. \quad (62)$$

Setting  $\Delta\Pi_x \approx \Delta\Pi_y \equiv \Delta\Pi$ , we get  $(\Delta\Pi)^2 \sim eB\hbar/c = m\hbar\omega_c$ . The spatial spread is  $\Delta r \sim \Delta\Pi/(m\omega_c) \sim \sqrt{\hbar c/(eB)} = \ell_B$ .

(3) **Classically.** The ground state energy is  $\frac{1}{2}\hbar\omega_c$ . Setting  $\frac{1}{2}mv^2 = \frac{1}{2}\hbar\omega_c$ :

$$v = \sqrt{\frac{\hbar\omega_c}{m}} = \sqrt{\frac{\hbar eB}{m^2 c}}, \quad R = \frac{mvc}{eB} = \frac{c}{eB} \sqrt{\frac{m\hbar eB}{c}} = \sqrt{\frac{\hbar c}{eB}} = \ell_B. \quad (63)$$

## Physical Insight

The parameter  $m$  in  $\Psi_0^{(m)}$  labels the angular momentum and controls the radius of the cyclotron orbit's guiding centre. For the same energy level,  $m$  changes the *shape* (and location) of the orbit. This becomes most transparent when we discuss the guiding centre in Section 9.

## 7 Quantum Mechanical Formulation: Landau Gauge

### 7.1 What Varies with Gauge Choice?

#### Note

Physics remains identical across gauges: all measurable quantities (energy levels, current, resistivity, ...) are gauge-invariant. Only the mathematical description of wave functions changes.

### 7.2 Setting up the Hamiltonian in Landau Gauge

Choose  $\vec{A} = (0, Bx, 0)$  (Landau gauge). Then:

$$\begin{aligned}
 H &= \frac{1}{2m} \left( \vec{p} - \frac{e\vec{A}}{c} \right)^2 + e\phi \\
 &= \frac{1}{2m} \left( \vec{p} - \frac{e\vec{A}}{c} \right) \cdot \left( \vec{p} - \frac{e\vec{A}}{c} \right) \\
 &= \frac{1}{2m} \left[ \left( p_x - \frac{eA_x}{c} \right)^2 + \left( p_y - \frac{eA_y}{c} \right)^2 + \left( p_z - \frac{eA_z}{c} \right)^2 \right] + e\phi(z) \\
 &= \frac{1}{2m} \left[ p_x^2 + \left( p_y - \frac{eBx}{c} \right)^2 + p_z^2 \right] + e\phi(z) \\
 &= \frac{1}{2m} \left[ p_x^2 + \left( p_y - \frac{eBx}{c} \right)^2 \right] + \left( \frac{p_z^2}{2m} + e\phi(z) \right) = H_{xy} + H_z. \quad (64)
 \end{aligned}$$

### 7.3 Conserved Quantities and Symmetry

Since  $H$  contains neither  $y$  nor  $z$  explicitly:

- $[p_z, H] = 0$  (no  $z$  in  $H$ ),  $[p_y, H] = 0$  (no  $y$  in  $H$ ).

If  $[O, H] = 0$ , then  $\langle O \rangle$  is conserved. Moving into simultaneous momentum eigenstates (from free particle):

$$\begin{aligned}
 H\psi &= E\psi, & p_y\psi &= \hbar k_y\psi, & \left[ \frac{p_z^2}{2m} - e\phi(z) \right] \psi &= E_z\psi, \\
 E &= E_{xy} + E_z, & H_z\psi &= E_z\psi.
 \end{aligned}$$

Since  $-i\hbar \frac{\partial \psi}{\partial y} = \hbar k_y\psi \Rightarrow \psi \propto e^{ik_y y}$ , the wave function factorises as:

$$\Psi(x, y, z) = e^{ik_y y} \phi(x) e^{ik_z z}, \quad E = E_{xy} + E_z, \quad E_z = \frac{\hbar^2 k_z^2}{2m}. \quad (65)$$

Substituting  $p_y\psi = \hbar k_y\psi$  into the full Schrödinger equation:

$$\begin{aligned} & \frac{1}{2m} \left[ p_x^2 + \left( \hbar k_y - \frac{eBx}{c} \right)^2 + p_z^2 \right] \psi - e\phi(z)\psi = E\psi \\ \Rightarrow & \frac{1}{2m} \left[ p_x^2 + \left( \hbar k_y - \frac{eBx}{c} \right)^2 \right] \phi(x) = (E - E_z) \phi(x). \end{aligned}$$

Noting that  $\hbar k_y - \frac{eBx}{c} = -\frac{eB}{c}(x - x_0)$  where  $x_0 = \frac{\hbar k_y c}{eB}$  is the **guiding centre**:

$$\Rightarrow \frac{1}{2m} \left[ p_x^2 + \frac{e^2 B^2}{c^2} (x - x_0)^2 \right] \phi(x) = (E - E_z) \phi(x).$$

$$\boxed{\frac{1}{2m} \left[ p_x^2 + \frac{e^2 B^2}{c^2} (x - x_0)^2 \right] \phi(x) = (E - E_z) \phi(x).} \quad (66)$$

This is precisely the 1D harmonic oscillator with frequency  $\omega_c = eB/mc$ , centred at  $x_0$ .

#### 7.4 Energy Eigenvalues and Wave Functions

Identifying  $\omega = \omega_c$  and  $\xi = x - x_0$ , the harmonic oscillator result gives:

$$E - E_z = \hbar\omega_c \left( n + \frac{1}{2} \right), \quad \Rightarrow \quad \boxed{E = \hbar\omega_c \left( n + \frac{1}{2} \right) + \frac{\hbar^2 k_z^2}{2m}.} \quad (67)$$

The ground state  $\phi_0$  is found from the lowering operator condition  $a\phi_0 = 0$ :

$$\begin{aligned} a = \sqrt{\frac{m\omega_c}{2\hbar}} \left( \xi + \frac{\hbar}{m\omega_c} \frac{d}{d\xi} \right), \quad a\phi_0 = 0 \Rightarrow & \left( \xi + \frac{\hbar}{m\omega_c} \frac{d}{d\xi} \right) \phi_0 = 0 \\ \frac{d\phi_0}{d\xi} = -\frac{m\omega_c}{\hbar} \xi \phi_0 \Rightarrow & \phi_0 = C \exp\left( -\frac{m\omega_c}{2\hbar} \xi^2 \right). \end{aligned}$$

Normalising:  $C^2 \int_{-\infty}^{\infty} \exp\left(-\frac{m\omega_c}{\hbar} \xi^2\right) d\xi = 1$ , so  $C = \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}$ .

The  $x$ -part of the wave function is:

$$\phi_0(x) = C \exp\left(-\frac{m\omega_c}{2\hbar}(x - x_0)^2\right), \quad C = \left(\frac{m\omega_c}{\pi\hbar}\right)^{1/4}. \quad (68)$$

The full ground-state wave function in Landau gauge is:

$$\boxed{\Psi_0(x, y, z) = \mathcal{N}_0 e^{ik_y y} \exp\left(-\frac{(x - x_0)^2}{2\ell_B^2}\right),} \quad (69)$$

describing a plane wave running along  $y$  and a Gaussian of width  $\ell_B$  localised around  $x_0$  — a “strip” running along  $y$ , localised around  $x_0$ .

## 7.5 Excited States

**Ground states in the symmetric gauge.** The ground Landau level ( $n = 0$ ) has many degenerate states labelled by the integer  $m$ :

$$|0, m\rangle \longrightarrow \Psi_g^{(m)} = \mathcal{N}_m z^m \exp\left(-\frac{|z|^2}{4\ell_B^2}\right), \quad m = 0, 1, 2, \dots \quad (70)$$

For *all* of these states,  $b|0, m\rangle = 0$ , and they all share the same energy  $E_0 = \frac{1}{2}\hbar\omega_c$ .

**Building excited states with  $b^\dagger$ .** Since  $b^\dagger$  raises energy, acting on any ground state generates a ladder of excited states:

$$|0, m\rangle \xrightarrow{b^\dagger} b^\dagger|0, m\rangle \xrightarrow{b^\dagger} (b^\dagger)^2|0, m\rangle \xrightarrow{b^\dagger} \dots \quad (71)$$

**Deriving the normalisation constant.** We need  $\langle n, m|n, m\rangle = 1$ . We use the key property of ladder operators: if  $b^\dagger b|\lambda\rangle = \lambda|\lambda\rangle$ , then  $(b^\dagger b)(b^\dagger|\lambda\rangle) = (\lambda + 1)b^\dagger|\lambda\rangle$ .

**For  $n = 1$ :** Compute the norm of  $b^\dagger|0, m\rangle$ :

$$(b^\dagger|0, m\rangle)^\dagger (b^\dagger|0, m\rangle) = \langle 0, m|bb^\dagger|0, m\rangle = \langle 0, m|(b^\dagger b + 1)|0, m\rangle = 1 = 1! \checkmark \quad (72)$$

**For  $n = 2$ :** Using  $b^\dagger b b^\dagger = b^\dagger(b^\dagger b + 1)$ :

$$(bb^\dagger)(b^\dagger|0\rangle) = (b^\dagger b + 1)b^\dagger|0\rangle = b^\dagger \underbrace{b^\dagger b|0\rangle}_{=0} + b^\dagger|0\rangle + b^\dagger|0\rangle = 2b^\dagger|0\rangle, \quad \text{so } b \cdot 2b^\dagger|0, m\rangle = 2 = 2! \checkmark \quad (73)$$

**General case (by induction):**

$$\langle 0, m|b^n (b^\dagger)^n|0, m\rangle = n! \quad (74)$$

Hence the normalisation constant is  $\sqrt{n!}$ :

### $n$ -th Excited Landau Level

$$|n, m\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}}|0, m\rangle, \quad H|n, m\rangle = \hbar\omega_c \left(n + \frac{1}{2}\right)|n, m\rangle. \quad (75)$$

**Explicit verification for  $n = 1$ .** Take  $|1, m\rangle = b^\dagger|0, m\rangle$ . Apply  $H = \hbar\omega_c(b^\dagger b + \frac{1}{2})$ :

$$b^\dagger b (b^\dagger|0, m\rangle) = b^\dagger(bb^\dagger)|0, m\rangle = b^\dagger(b^\dagger b + 1)|0, m\rangle = b^\dagger|0, m\rangle = |1, m\rangle. \quad (76)$$

So  $b^\dagger b|1, m\rangle = |1, m\rangle$ , giving  $E_1 = \hbar\omega_c(1 + \frac{1}{2}) = \frac{3}{2}\hbar\omega_c$ . ✓

**Verification for  $n = 2$ .**  $|2, m\rangle = \frac{(b^\dagger)^2}{\sqrt{2}}|0, m\rangle$ , giving  $E_2 = \hbar\omega_c(2 + \frac{1}{2}) = \frac{5}{2}\hbar\omega_c$ .

**Every Landau level has the same degeneracy.** The index  $m$  runs over the degenerate ground states; applying  $b^\dagger$  does not change  $m$ . Every Landau level  $n$  has the full set of degenerate states  $\{|n, m\rangle : m = 0, 1, 2, \dots\}$ .

## 7.6 Angular Momentum with Ladder Operators

In the symmetric gauge there are in fact *two* independent pairs of kinetic momenta:

$$\Pi_x = p_x + \frac{eB}{2c}y, \quad \Pi_y = p_y - \frac{eB}{2c}x \quad (\text{cyclotron, enter } H), \quad (77)$$

$$\tilde{\Pi}_x = p_x - \frac{eB}{2c}y, \quad \tilde{\Pi}_y = p_y + \frac{eB}{2c}x \quad (\text{guiding centre, commute with } H). \quad (78)$$

Their commutators:

$$[\Pi_x, \Pi_y] = +\frac{ieB\hbar}{c}, \quad [\tilde{\Pi}_x, \tilde{\Pi}_y] = -\frac{ieB\hbar}{c}, \quad [\Pi_i, \tilde{\Pi}_j] = 0 \quad \forall i, j. \quad (79)$$

From these we define two pairs of ladder operators:

$$b = \sqrt{\frac{c}{2eB\hbar}}(\Pi_x + i\Pi_y), \quad b^\dagger = \sqrt{\frac{c}{2eB\hbar}}(\Pi_x - i\Pi_y), \quad (80)$$

$$a = \sqrt{\frac{c}{2eB\hbar}}(\tilde{\Pi}_x - i\tilde{\Pi}_y), \quad a^\dagger = \sqrt{\frac{c}{2eB\hbar}}(\tilde{\Pi}_x + i\tilde{\Pi}_y). \quad (81)$$

Because  $[\Pi_i, \tilde{\Pi}_j] = 0$ , all cross-commutators vanish:

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1, \quad [a, b] = 0, \quad [a^\dagger, b^\dagger] = 0, \quad [a^\dagger, b] = 0, \quad [a, b^\dagger] = 0. \quad (82)$$

**Role of the  $a$  operators.** The Hamiltonian  $H = \hbar\omega_c(b^\dagger b + \frac{1}{2})$  depends only on  $b, b^\dagger$ , so:

$$[H, a] = 0, \quad [H, a^\dagger] = 0. \quad (83)$$

Thus  $a^\dagger a$  is a **conserved quantity** that labels the degeneracy:

$$a^\dagger a |\tilde{m}\rangle = \tilde{m} |\tilde{m}\rangle, \quad \tilde{m} = 0, 1, 2, \dots \quad (84)$$

**Connection to angular momentum.** The  $z$ -component of orbital angular momentum is:

$$L_z = \hbar(a^\dagger a - b^\dagger b) = \hbar(\tilde{m} - n). \quad (85)$$

So  $\tilde{m}$  and  $n$  are the angular momentum quantum numbers of the problem.

**Derivation of the  $L_z$  formula.** Starting from  $L_z = xp_y - yp_x$  and inverting the kinetic momenta:

$$p_x = \frac{1}{2}(\Pi_x + \tilde{\Pi}_x), \quad y = \frac{c}{eB}(\Pi_x - \tilde{\Pi}_x), \quad (86)$$

$$p_y = \frac{1}{2}(\Pi_y + \tilde{\Pi}_y), \quad x = \frac{c}{eB}(\tilde{\Pi}_y - \Pi_y). \quad (87)$$

Substituting into  $L_z$  and using  $[\Pi_i, \tilde{\Pi}_j] = 0$  so products simplify:

$$\begin{aligned} L_z &= \frac{c}{eB}(\tilde{\Pi}_y - \Pi_y) \cdot \frac{1}{2}(\Pi_y + \tilde{\Pi}_y) - \frac{c}{eB}(\Pi_x - \tilde{\Pi}_x) \cdot \frac{1}{2}(\Pi_x + \tilde{\Pi}_x) \\ &= \frac{c}{2eB} \left[ (\tilde{\Pi}_x^2 + \tilde{\Pi}_y^2) - (\Pi_x^2 + \Pi_y^2) \right]. \end{aligned} \quad (88)$$

Now using  $\Pi_x^2 + \Pi_y^2 = \frac{2eB\hbar}{c}(b^\dagger b + \frac{1}{2})$  and  $\tilde{\Pi}_x^2 + \tilde{\Pi}_y^2 = \frac{2eB\hbar}{c}(a^\dagger a + \frac{1}{2})$ :

$$L_z = \frac{c}{2eB} \cdot \frac{2eB\hbar}{c} \left[ \left( a^\dagger a + \frac{1}{2} \right) - \left( b^\dagger b + \frac{1}{2} \right) \right] = \hbar(a^\dagger a - b^\dagger b). \quad (89)$$

### Key Result

$$L_z = \hbar(a^\dagger a - b^\dagger b) = \hbar(\tilde{m} - n). \quad \checkmark \quad (90)$$

The half-integer shifts cancel exactly, leaving a clean integer-valued angular momentum.

**Complete basis of states.** The full two-index state is built from the double vacuum  $b|0, 0\rangle = 0$ ,  $a|0, 0\rangle = 0$ :

$$|n, \tilde{m}\rangle = \frac{(b^\dagger)^n}{\sqrt{n!}} \cdot \frac{(a^\dagger)^{\tilde{m}}}{\sqrt{\tilde{m}!}} |0, 0\rangle, \quad H|n, \tilde{m}\rangle = \hbar\omega_c \left( n + \frac{1}{2} \right) |n, \tilde{m}\rangle. \quad (91)$$

Here  $b^\dagger$  steps up the Landau level (raises energy), while  $a^\dagger$  steps within the degenerate subspace of a given level (changes  $\tilde{m}$ , i.e. the location of the guiding centre), *without changing the energy*.

## 8 Degeneracy of Landau Levels

### 8.1 Counting States in the Landau Gauge

Take a finite rectangular sample of dimensions  $L_x \times L_y$  and set  $k_z = 0$ .

The wave function  $\Psi \propto e^{ik_y y} \exp(-(x - x_0)^2/2\ell_B^2)$  is periodic in  $y$ :

$$\Psi(x, y + L_y) = \Psi(x, y) \implies k_y = \frac{2\pi n_y}{L_y}, \quad n_y = 0, \pm 1, \pm 2, \dots \quad (92)$$

The guiding centre must lie within the sample:  $0 \leq x_0 \leq L_x$ , i.e.

$$0 \leq \frac{\hbar k_y c}{eB} \leq L_x \implies 0 \leq k_y \leq \frac{eBL_x}{\hbar c}. \quad (93)$$

Number of allowed  $k_y$  values:

$$N = \frac{(k_y)_{\max}}{2\pi/L_y} = \frac{eBL_x}{\hbar c} \cdot \frac{L_y}{2\pi} = \frac{eB L_x L_y}{2\pi\hbar c} = \frac{BA}{\Phi_0}, \quad (94)$$

where  $A = L_x L_y$  is the sample area and  $\Phi_0 = hc/e$  is the **magnetic flux quantum**.

### Degeneracy

$$N = \frac{BA}{\Phi_0} = \frac{eBA}{hc}. \quad (95)$$

Each Landau level is degenerate with  $N$  states per level, equal to the number of flux quanta threading the sample. This result is *gauge-independent* and *sample-size-dependent*.

## 8.2 Degeneracy in Symmetric Gauge

In the symmetric gauge, the ground state degeneracy is labelled by  $m = 0, 1, 2, \dots$ . For a finite sample of area  $A = \pi R_s^2$ , the guiding centre must lie inside:  $R_{gc}^2 \leq R_s^2$ . The guiding-centre radius-squared operator is:

$$R_{gc}^2 = 2\ell_B^2 \left( \tilde{m} + \frac{1}{2} \right), \quad (96)$$

where  $\tilde{m}$  is the eigenvalue of  $a^\dagger a$  (see Section 7.6). The constraint  $R_{gc}^2 \leq R_s^2$  gives:

$$\tilde{m} \lesssim \frac{R_s^2}{2\ell_B^2} \sim \frac{eBA}{\pi \cdot 2\hbar c}, \quad (97)$$

and  $N = \tilde{m}_{\max} + 1 \approx \frac{eBA}{\hbar c} = \frac{BA}{\Phi_0}$ , same result as Landau gauge.

## 9 The Guiding Centre Algebra

### 9.1 Classical Guiding Centre

Classically, an electron in a magnetic field traces a circular orbit. The *guiding centre*  $(\mathcal{X}, \mathcal{Y})$  is the centre of that orbit:

$$x(t) = \mathcal{X} + R \cos(\omega_c t), \quad y(t) = \mathcal{Y} + R \sin(\omega_c t). \quad (98)$$

From the equations of motion:

$$\dot{y} = \omega_c(x - \mathcal{X}) \Rightarrow \mathcal{Y} = y + \frac{\dot{x}}{\omega_c}. \quad (99)$$

Using canonical variables in the symmetric gauge, the guiding centre coordinates are:

$$\mathcal{X} = x - \frac{c}{eB}\Pi_y, \quad \mathcal{Y} = y + \frac{c}{eB}\Pi_x. \quad (100)$$

## 9.2 Quantum Guiding Centre Operators

Promoting to operators:

$$\hat{\mathcal{X}} = x + \frac{c}{eB}\Pi_y, \quad \hat{\mathcal{Y}} = y - \frac{c}{eB}\Pi_x. \quad (101)$$

Their commutator:

$$[\hat{\mathcal{X}}, \hat{\mathcal{Y}}] = -i\ell_B^2, \quad (102)$$

which is another canonical pair (after rescaling). The second pair of ladder operators  $a, a^\dagger$  (defined in Section 7.6) generate this algebra. These satisfy:

$$[a, a^\dagger] = 1, \quad [a, b] = 0, \quad [a^\dagger, b^\dagger] = 0, \quad [a, b^\dagger] = 0. \quad (103)$$

Since  $[H, a] = 0$ ,  $a^\dagger a$  is conserved and labels the degeneracy:  $\tilde{m} = 0, 1, 2, \dots$  gives the location of the guiding centre within a given Landau level.

**Angular momentum connection.** From the derivation in Section 7.6:  $L_z = \hbar(\tilde{m} - n)$ . So  $\tilde{m}$  and  $n$  are the angular momentum quantum numbers of the problem.

**Radius of guiding centre.**

$$R_{gc}^2 = \mathcal{X}^2 + \mathcal{Y}^2 = \frac{c^2}{e^2 B^2} (\tilde{\Pi}_x^2 + \tilde{\Pi}_y^2) = 2\ell_B^2 \left( \tilde{m} + \frac{1}{2} \right). \quad (104)$$

The guiding centre radius grows with  $\tilde{m}$ ;  $n$  and  $\tilde{m}$  are independent, confirming that the degeneracy is the same for every Landau level.

**General state and Hamiltonian.** For the Hamiltonian  $H = \hbar\omega_c(b^\dagger b + \frac{1}{2})$ :

$$[H, a] = 0, \quad [H, a^\dagger] = 0 \Rightarrow a^\dagger a \text{ labels degeneracy, } a^\dagger a |\tilde{m}\rangle = \tilde{m} |\tilde{m}\rangle. \quad (105)$$

The full state is  $|n, \tilde{m}\rangle$  with energy  $\hbar\omega_c(n + \frac{1}{2})$ , independent of  $\tilde{m}$ .

## 10 Hall Effect

### 10.1 Classical Hall Effect

Consider a 2D electron gas in crossed fields:  $\vec{E} = E_x \hat{x}$  (applied),  $\vec{B} = B \hat{z}$  (perpendicular). Positive charges are deflected in the  $-y$  direction by the magnetic force:

$$\vec{F}_{\text{mag}} = \frac{q}{c} \vec{v} \times \vec{B} = \frac{qv_x}{c} (\hat{x} \times \hat{z}) = -\frac{qv_x B}{c} \hat{y}. \quad (106)$$

Charge accumulates at the edges, generating a transverse electric field  $E_y$  (the Hall field). At equilibrium, the electric and magnetic forces balance:

$$qE_y = \frac{qvB}{c} \implies \boxed{V_H = E_y w = \frac{vBw}{c}}, \quad (107)$$

where  $w$  is the sample width. This is the **Hall voltage**.

### 10.2 Hall Resistance

The Hall resistance is defined as the ratio of Hall voltage to current (transverse to voltage):

$$R_{xy} = \frac{V_H}{I_x}, \quad (108)$$

whereas the ordinary (longitudinal) resistance is  $R_{xx} = V_x/I_x$ .

In 3D, the current density is  $j_x = qnv$ , so  $I_x = j_x(wt) = qnvwt$ :

$$R_{xy} = \frac{wvB/c}{qnvwt} = \frac{B}{qnct} \implies \rho_{xy} = R_{xy} \cdot t = \frac{B}{cq n} \quad (3\text{D resistivity}). \quad (109)$$

where  $t$  is the sample thickness.

In 2D ( $j_x^{(2D)} = qnv$ ,  $I_x = j_x^{(2D)} w = qnvw$ ):

$$R_{xy} = \frac{vBw/c}{qnvw} = \frac{B}{cq n} \quad (110)$$

### 10.3 Longitudinal Resistance

The longitudinal resistance is defined as:

$$R_{xx} = \frac{V_x}{I_x}. \quad (111)$$

**3D Case:** In three dimensions, the current density is  $j_x = I_x/A$ , where  $A$  is the cross-sectional area. Using  $E_x = \rho_{xx} j_x$ , we obtain:

$$R_{xx} = \rho_{xx} \frac{L_x}{A}, \quad (112)$$

where  $L_x$  is the length of the sample.

**2D Case (Quantum Hall System):** For a two-dimensional electron gas, current flows in a plane and  $I_x = j_x w$ , where  $w$  is the sample width. Thus:

$$R_{xx} = \rho_{xx} \frac{L_x}{w}, \quad (113)$$

where  $w$  is the width of the sample.

#### Note

In Quantum Hall systems, the 2D expression is relevant. In the quantum Hall limit,  $\rho_{xx} \rightarrow 0$ , leading to vanishing longitudinal resistance despite finite Hall resistance.

### 10.4 $E \times B$ Drift

Without scattering, the equation of motion at steady state ( $m\dot{\vec{v}} = 0$ ):

$$0 = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B} \Rightarrow \vec{E} = -\frac{1}{c}\vec{v}_{\text{drift}} \times \vec{B}. \quad (114)$$

Cross both sides with  $\vec{B}$ :

$$\vec{E} \times \vec{B} = -\frac{1}{c}(\vec{v}_{\text{drift}} \times \vec{B}) \times \vec{B} = -\frac{1}{c}[(\vec{v}_{\text{drift}} \cdot \vec{B})\vec{B} - B^2\vec{v}_{\text{drift}}] = \frac{B^2}{c}\vec{v}_{\text{drift}} \quad (\text{since } \vec{v}_{\text{drift}} \perp \vec{B}). \quad (115)$$

$$\vec{v}_{\text{drift}} = \frac{c}{B^2}(\vec{E} \times \vec{B}). \quad (116)$$

With  $\vec{E} = E_x \hat{x}$  and  $\vec{B} = B \hat{z}$ :  $\vec{v}_{\text{drift}} = -\frac{cE_x}{B} \hat{y}$  (no drift along  $x$ ), so  $j_x = 0$  and  $\rho_{xx} = E_x/j_x \rightarrow \infty$ .

#### Physical Insight

Without scattering, the magnetic force does no work ( $\vec{F}_{\text{mag}} \cdot \vec{v} = 0$ ), so it cannot dissipate energy and hence cannot create longitudinal resistance:  $\rho_{xx} = 0$ . The majority of the current is the Hall current.

## 11 Conductivity Tensor and Onsager Relations

### 11.1 The Conductivity Tensor

For an ordinary material under a magnetic field, the current-field relation generalises to a  $2 \times 2$  tensor:

$$\begin{pmatrix} j_x \\ j_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \quad (117)$$

**Symmetry constraints.**

- (i) **Rotational symmetry about  $\hat{z}$ :** The magnetic field lies along  $z$ , so effects are conserved when rotated about  $z$ . This forces  $\sigma_{xx} = \sigma_{yy}$ .
- (ii) **Onsager reciprocal relation (time-reversal):** Under

$$t \rightarrow -t, \quad \vec{E} \rightarrow \vec{E}, \quad \vec{B} \rightarrow -\vec{B}, \quad \vec{j} \rightarrow -\vec{j}, \quad \vec{v} \rightarrow -\vec{v}.$$

The Onsager reciprocal relation gives  $\sigma_{ij}(B) = \sigma_{ji}(-B)$ . Combined with the inversion argument:  $\sigma_{yx}(B) = -\sigma_{xy}(B)$ , i.e.  $\sigma_{yx} = -\sigma_{xy}$ .

Therefore:

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}. \quad (118)$$

where  $\sigma_{xx}$  is the longitudinal conductivity and  $\sigma_{xy}$  is the Hall conductivity.

## 11.2 Resistivity Tensor

The resistivity tensor is  $\rho = \sigma^{-1}$ . For a  $2 \times 2$  matrix:

$$\rho = \sigma^{-1} = \frac{1}{\sigma_{xx}^2 + \sigma_{xy}^2} \begin{pmatrix} \sigma_{xx} & -\sigma_{xy} \\ \sigma_{xy} & \sigma_{xx} \end{pmatrix}, \quad (119)$$

giving:

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}, \quad \rho_{xy} = -\frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2}. \quad (120)$$

**Perfect conductor limit ( $\sigma_{xx} \rightarrow 0$ ):**  $\rho_{xx} \rightarrow 0$  and  $\rho_{xy} \rightarrow -1/\sigma_{xy}$ . This looks contradictory (acts as both perfect conductor and insulator), but  $\rho_{xy} \neq 0$  means it acts as an insulator in the longitudinal direction. The material carries current purely transversely (Hall current).

# 12 Integer Quantum Hall Effect (IQHE)

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## 12.1 Discovery and Setup

The IQHE (von Klitzing, 1980) considers a 2D electron gas in a plane, subject to a perpendicular magnetic field.

- The chemical potential  $\mu = \frac{\partial F}{\partial N}$  (free energy per electron).
- At low temperature  $T$ ,  $\mu \approx E_F$  (**Fermi energy** : highest energy level occupied by electrons in a material at absolute zero temperature).
- Classically,  $\rho_{xy} = \frac{B}{cen}$  grows linearly with  $B$ .

## 12.2 Landau Level Filling

For strong magnetic fields, the spectrum collapses into discrete Landau levels:

$E_0 = \frac{1}{2}\hbar\omega_c$ ,  $E_1 = \frac{3}{2}\hbar\omega_c$ ,  $\dots$ , each holding  $\frac{N}{A} = \frac{\frac{BA}{\phi_0}}{A} = \frac{B}{\phi_0} = \frac{B}{\frac{hc}{e}} = \frac{eB}{hc}$  electrons per unit area.

If exactly  $\nu$  Landau levels are filled at low temperature (and higher levels empty):

Level( $n$ )	Energy	States/area
$n = 0$	$\frac{1}{2}\hbar\omega_c$	$eB/hc$ (filled)
$n = 1$	$\frac{3}{2}\hbar\omega_c$	$eB/hc$ (filled)
$\vdots$	$\vdots$	$\vdots$
$n = \nu - 1$	$(\nu - \frac{1}{2})\hbar\omega_c$	$eB/hc$ (filled)
$n = \nu$	$(\nu + \frac{1}{2})\hbar\omega_c$	$eB/hc$ (empty)

Total electron density:  $n_e = \nu \cdot \frac{eB}{hc}$ .

Once a set of Landau levels is completely filled, the Fermi level jumps to the next available level. This “jumping” quantises the Hall conductance.

### 12.3 Quantised Hall Resistivity

Substituting  $n_e = \frac{\nu eB}{hc}$  into  $\rho_{xy} = \frac{B}{cen_e}$ :

$$\rho_{xy} = \frac{B}{c \cdot e \cdot \frac{\nu eB}{hc}} = \frac{B \cdot hc}{c \cdot \nu e^2 B} = \frac{h}{\nu e^2}. \quad (121)$$

#### IQHE Result

$$\rho_{xy} = \frac{h}{\nu e^2}, \quad \nu = 1, 2, 3, \dots \quad (122)$$

The result depends only on  $h$ ,  $e$ , and the integer  $\nu$  — *not* on  $B$ , the material, impurities, or sample geometry. Therefore the quantity  $\frac{h}{e^2}$  can be interpreted as the quantum of resistivity and is known as the von Klitzing constant.

The key point: the result depends only on fundamental constants and the integer  $\nu$  (the material parameters cancel entirely).

## 13 Beyond IQHE: Fractional Quantum Hall Effect

### 13.1 Basic Introduction

The FQHE (Tsui & Störmer, 1982) observes Hall plateaux at *fractional* fillings:

$$\rho_{xy} = \frac{h}{fe^2}, \quad f = \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{5}, \dots \quad (123)$$

IQHE fails to include *electron-electron interactions*. Real electrons repel each other via the Coulomb interaction  $V = \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$ .

For a *partially filled* Landau level (e.g.  $\nu = 1/3$ ), electrons arrange themselves among available states to minimise Coulomb repulsion. This creates an *energy gap* to the next excited state — not a Landau level gap, but one arising from correlations between electrons. These interaction-induced energy gaps prevent scattering and create a plateau.

### 13.2 Laughlin Wave Function

Laughlin (1983) proposed a trial wave function for  $\nu = 1/m$  ( $m$  odd):

$$\Psi_m(z_1, z_2, \dots, z_N) = \prod_{i < j} (z_i - z_j)^m \exp\left(-\sum_i \frac{|z_i|^2}{4\ell_B^2}\right), \quad (124)$$

where  $z_i = x_i + iy_i$  is the complex position of the  $i$ -th electron,  $m$  is an odd integer and  $\ell_B = \sqrt{\frac{\hbar c}{eB}}$  is the magnetic length.

**Jastrow factor.** The product  $\prod_{i < j} (z_i - z_j)^m$  is the *Jastrow factor*, which ensures that when two electrons come close ( $z_i \approx z_j$ ),  $\Psi \rightarrow 0$ . This keeps electrons maximally separated.

**Why  $m$  must be odd?** Electrons are fermions; therefore the wave function must be anti-symmetric under exchange of particles  $i$  and  $j$ :

$$\Psi(\dots, z_i, \dots, z_j, \dots) = -\Psi(\dots, z_j, \dots, z_i, \dots). \quad (125)$$

Under exchange:  $(z_j - z_i)^m = (-1)^m (z_i - z_j)^m$ . For anti-symmetry,  $(-1)^m = -1$ , so  $m$  must be **odd**.

#### Key properties:

- The Gaussian  $\exp(-\sum_i |z_i|^2/4\ell_B^2)$  is the same as in the Landau ground state.
- Multiple values of  $m$  give the same energy (IQHE ground-state degeneracy analogue).
- Excitations above the  $\nu = 1/m$  state carry *fractional charge*  $e/m$ .
- **Particle-hole symmetry:**  $\nu = 2/3$  is equivalent to  $\nu = 1/3$  *with holes* (i.e. with electrons absent). The two states are related by exchanging particles and holes, a symmetry that produces a mirror structure in the FQHE plateau sequence.

## 14 Summary and Key Formulae

Symbol	Expression	Name
$\omega_c$	$eB/mc$	Cyclotron frequency
$\ell_B$	$\sqrt{\hbar c/eB}$	Magnetic length
$E_n$	$\hbar\omega_c(n + \frac{1}{2})$	Landau levels
$N/A$	$eB/hc$	Degeneracy per area
$\Phi_0$	$hc/e$	Flux quantum
$\nu$	$nhc/(eB)$	Filling factor
$\rho_{xy}$ (IQHE)	$h/\nu e^2$	Quantised Hall resistivity
$\rho_{xy}$ (FQHE)	$h/fe^2$	Fractional Hall resistivity
$\Psi_0^{(m)}$	$\mathcal{N}_m z^m e^{- z ^2/4\ell_B^2}$	Ground state (symm. gauge)
$\Psi_0$ (Landau)	$e^{iky} e^{-(x-x_0)^2/2\ell_B^2}$	Ground state (Landau gauge)
$\Psi_m$ (Laughlin)	$\prod_{i<j} (z_i - z_j)^m e^{-\sum  z_i ^2/4\ell_B^2}$	Laughlin state

### 14.1 Conceptual Summary

1. **Classical mechanics** gives cyclotron orbits and establishes  $H = (\vec{p} - e\vec{A}/c)^2/2m$ .
2. **Gauge freedom:**  $\vec{A} \rightarrow \vec{A} + \nabla\chi$  leaves  $\vec{B}$  unchanged; physics is gauge-invariant.
3. **Kinetic momentum operators**  $\Pi_x, \Pi_y$  satisfy  $[\Pi_x, \Pi_y] = ieB\hbar/c$ , enabling ladder operators identical to the harmonic oscillator.
4. **Landau levels**  $E_n = \hbar\omega_c(n + \frac{1}{2})$  are the quantised energies; each level is massively degenerate ( $N = BA/\Phi_0$ ).
5. **Guiding centre** coordinates  $(\mathcal{X}, \mathcal{Y})$  commute with  $H$  and label the degeneracy independently of  $n$ .
6. **Filling  $\nu$  levels** gives carrier density  $n = \nu eB/hc$ , and substitution into the classical Hall formula yields the *quantised*  $\rho_{xy} = h/\nu e^2$ .
7. **FQHE** requires electron correlations; Laughlin's wave function with odd integer  $m$  describes the  $\nu = 1/m$  plateau.

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